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On Some Generalised Transmuted Distributions

Kishore K. Das
Luku Deka Barman
Department of Statistics
Gauhati University

Abstract

In this paper a generalized form of the transmuted distributions has been defined. Their moments and other distributional properties of this distribution has been studied. Finally deriving transmuted skew exponential distribution its different distributional properties has been discussed.

1 Introduction

The transmutation map approach has been suggested by Shaw and Buckley [3], which has been laid down in the following.

1.1 Rank Transmutation

Suppose that we have two distributions with common sample space, with CDFs F_1 and F_2. Then under rank transmutation, we can form (Shaw and Buckley [3])

$$G_{R12}(u) = F_2(F_1^{-1}(u)), \quad G_{R21}(u) = F_1(F_2^{-1}(u)) \tag{1.1}$$

This pair of maps takes unit interval $I = (0, 1)$ into itself and under suitable assumptions are mutual inverse and satisfy $G_{ij}(0) = 0$ and $G_{ij}(1) = 1$. It is also assume that these rank transmutation maps are continuously differentiable.

1.2 Quadratic Transmutation

Under Quadratic transmutation the CDF of Skewed distribution can be obtained by using the relation of CDF of base distribution as (Shaw and Buckley [3])

$$F_2(x) = (1 + \lambda)\ F_1(x) - \lambda(F_1(x))^2 \tag{1.2}$$

Where $F_1(x)$ is the CDF of base distribution and $|\lambda| \leq 1$, known as shape parameter.

The transmutation map approach has been suggested by Shaw and Buckley [3] to define a new model which generalizes the skew exponential distribution. We will call the generalized distribution as the transmuted skew exponential distribution. According to the Quadratic Rank Transmutation Map (QRTM), approach the cumulative distribution function (cdf) satisfy the relationship

$$F(x) = (1+\alpha)\ G(x) - \alpha\ (G(x))^2, \quad \alpha \leq 1 \tag{1.3}$$

where $G(x)$ is the cdf of the base distribution. It is observe that at $\alpha = 0$, the cdf (1.3) becomes the distribution function of the base random variable. One can express the cdf (1.3) in terms of probability density function (pdf) and is given by (see Johnson et al. [2])

$$f(x;\alpha) = (1+\alpha)\ g(x) - 2\ \alpha\ g(x)G(x) \tag{1.4}$$

The standard exponential distribution has the probability density function (pdf) and the cumulative distribution function (cdf) specified by

$$f(x) = \exp(-x), \quad x > 0, \tag{1.5}$$

and

$$F(x) = 1 - \exp(-x) \tag{1.6}$$

respectively, where $x > 0$. A random variable X is said to have Azzalinis skew-exponential distribution if its pdf is

$$f(x,\alpha) = K\, g(x)G(\alpha x), \tag{1.7}$$

where $K^{-1} = \int_0^\infty g(x)G(\alpha\, x)dx$, $x \in R$ and $\alpha \in R$ (Azzalini [1]). The pdf (1.5) can be defined as

Definition 1 *A random variable X is said to have a skew exponential distribution, if it has the following pdf*

$$f(x,\alpha) = \frac{(\alpha+1)\ e^{-x}\ (1-e^{-\alpha\, x})}{\alpha}, \quad x \geq 0,\ \alpha \neq 0 \tag{1.8}$$

In this paper a generalized form of the transmuted distributions has been defined. Their moments and other distributional properties of this distribution has been studied. Next transmuted skew exponential distribution has been derived and finally, its different distributional properties has been discussed.

2 Generalized Transmuted Distribution

The generalized form of cumulative distribution function of the transmuted distribution is defined by

Definition 2 *A random variable is said to follow a generalized transmuted distribution (GTD) function, if it has the following cdf*

$$F(x) = \left(1 + \sum_{j=1}^{k} \alpha_j\right)\ G(x) - \sum_{j=1}^{k} \alpha_j\ (G(x))^2, \tag{2.9}$$

where $G(x)$ is the cdf of the base distribution, and α_j, $j = 1, \ldots, k$ are the constants to be estimated.

The probability density function of the generalized transmuted distribution (GTD) is obtained as

Definition 3 *A random variable is said to follow a generalized transmuted distribution function if it has the following pdf*

$$f(x) = \left(1 + \sum_{j=1}^{k} \alpha_j\right) \; g(x) - 2\sum_{j=1}^{k} \alpha_j \; g(x)G(x), \tag{2.10}$$

where $G(x)$ is the cdf of the base distribution, $g(x)$ is the pdf of the base distribution and α_j, $j = 1, \ldots, k$ are the constants to be estimated.

3 Moment Generating Function of the Transmuted Distribution

Let the random variable X_g and X_h having the pdfs' $g(x)$ and $h(x) = 2g(x)G(x)$ respectively. Then the moment generating function of the random variables are denoted by $M_{X_g}(t)$ and $M_{X_h}(t)$ respectively. Now we define the moment generating function (mgf) of the GTD as

Definition 4 *Let the random variable X follows GTD with pdf (2.10), then the mgf of GTD variate is given by*

$$M_{X_f}(t) = \left(1 + \sum_{j=1}^{k} \alpha_j\right) \; M_{X_g}(t) - 2 \; \sum_{j=1}^{k} \alpha_j \; M_{X_h}(t) \tag{3.11}$$

where the random variables X_g and X_h having the pdf $g(x)$ and $h(x) = 2g(x)G(x)$ respectively.

3.1 Moments of the Transmuted Distribution

The rth moments about origin of the random variables X_f, X_g and X_h are denoted by $\mu'_{r:f}$, $\mu'_{r:g}$ and $\mu'_{r:h}$ respectively. The rth moment about origin of the X_f is obtained differentiating (3.11) r times with respect to t and putting them for $t = 0$ and given by

$$\begin{aligned} \frac{d}{dt} M_{X_f}(t)|_{t=0} &= \left(1 + \sum_{j=1}^{k} \alpha_j\right) \; M_{X_g}(t)|_{t=0} - 2 \; \sum_{j=1}^{k} \alpha_j \; M_{X_h}(t)|_{t=0} \\ \mu'_{r:f} &= \left(1 + \sum_{j=1}^{k} \alpha_j\right) \; \mu'_{r:g} - 2 \; \sum_{j=1}^{k} \alpha_j \; \mu'_{r:h} \end{aligned} \tag{3.12}$$

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4 Characteristics Function of the Transmuted Distribution

Let the random variable X_g and X_h having the pdfs' $g(x)$ and $h(x) = 2g(x)G(x)$ respectively. Then the characteristic function of the random variables are denoted by $\Phi_{X_g}(t)$ and $\Phi_{X_h}(t)$ respectively. Then we define the characteristic function (cf) of the GTD as

Definition 5 *Let the random variable X follows GTD with pdf (2.10), then the cf of GTD variate is given by*

$$\Phi_{X_f}(t) = \left(1 + \sum_{j=1}^{k} \alpha_j\right) \ \Phi_{X_g}(t) - 2 \ \sum_{j=1}^{k} \alpha_j \ \Phi_{X_h}(t) \tag{4.13}$$

where the random variables X_g and X_h having the pdf $g(x)$ and $h(x) = 2g(x)G(x)$ respectively.

4.1 Moments from Characteristic Function

The rth moments about origin of the random variables X_f, X_g and X_h are denoted by $\mu'_{r:f}$, $\mu'_{r:g}$ and $\mu'_{r:h}$ respectively. The rth moment about origin of the X_f is obtained by differentiating (4.13) r times with respect to t and putting them for $t = 0$ and given by

$$\begin{aligned} \frac{d^r}{dt^r} \Phi_{X_f}(t)|_{t=0} &= \left(1 + \sum_{j=1}^{k} \alpha_j\right) \ \Phi_{X_g}(t)|_{t=0} - 2 \ \sum_{j=1}^{k} \alpha_j \ \Phi_{X_h}(t)|_{t=0} \\ \mu'_{r:f} &= \left(1 + \sum_{j=1}^{k} \alpha_j\right) \ \mu'_{r:g} - 2 \ \sum_{j=1}^{k} \alpha_j \ \mu'_{r:h} \qquad (4.14) \end{aligned}$$

5 Transmuted Skew Exponential Distribution

Using the quadratic rank transmutation map (1.4), we define the transmuted skew exponential distribution with parameter α as

Definition 6 *Let the random variable X is said to follow transmuted skew exponential distribution if it has the following pdf*

$$f(x;\alpha) = \begin{cases} \frac{(\alpha+1)\,(1-e^{\alpha\,x})\,(((\alpha-1)\,e^{x}-2\,\alpha-2)\,e^{\alpha\,x}+2)\,e^{-2\,\alpha\,x-2\,x}}{\alpha}, & \alpha \neq 1,\ x > 0 \\ \frac{(\alpha+1)\left(1-e^{-x}\,(x\,e^{x}+1)\right)}{\alpha} & \alpha = 1,\ x > 0 \end{cases} \tag{5.15}$$

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The cdf of the pdf (5.15) is given by

$$G(x;\alpha)=\begin{cases}\frac{e^{-2\,\alpha\,x-2\,x}\left(e^{\alpha\,x+x}+\alpha\,e^{\alpha\,x}+e^{\alpha\,x}-1\right)\left(\alpha\,e^{\alpha\,x+x}-\alpha\,e^{\alpha\,x}-e^{\alpha\,x}+1\right)}{\alpha}, & -\frac{2}{\alpha}-3\neq-1,-\frac{2}{\alpha}-2\neq-1,\ -\frac{2}{\alpha}-2\neq-1,\ -\frac{1}{\alpha}-2\neq-1,\ -\frac{1}{\alpha}-2\neq-1\\ -\frac{(\alpha+1)\left(e^{-2\,x}\left((\alpha+3)\,x\,e^{2\,x}+(1-\alpha)\,e^{x}+\alpha+1\right)-2\right)}{\alpha} & -\frac{2}{\alpha}-3=-1,-\frac{2}{\alpha}-2=-1,\ -\frac{2}{\alpha}-2=-1,\ -\frac{1}{\alpha}-2=-1,\ -\frac{1}{\alpha}-2=-1\end{cases} \quad (5.16)$$

6 Moments

In the following we derive first four moments about origin, i.e., first four raw moments. First, we find first raw moment, i.e., mean of the distribution as

$$\begin{aligned}\text{Mean}=E(X) &= -\frac{(\alpha+1)\,(\alpha^4+\alpha^3-5\,\alpha^2-8\,\alpha)}{\alpha\,(2\,\alpha^3+8\,\alpha^2+10\,\alpha+4)}\\ &= -\frac{\alpha^3+\alpha^2-5\,\alpha-8}{2\,(\alpha+1)\,(\alpha+2)}\end{aligned} \quad (6.17)$$

Next we find the second raw moment

$$\begin{aligned}E(X^2) &= -\frac{(\alpha+1)\,(3\,\alpha^6+16\,\alpha^5+22\,\alpha^4-16\,\alpha^3-66\,\alpha^2-48\,\alpha)}{\alpha\,(2\,\alpha^5+14\,\alpha^4+38\,\alpha^3+50\,\alpha^2+32\,\alpha+8)}\\ &= -\frac{3\,\alpha^5+16\,\alpha^4+22\,\alpha^3-16\,\alpha^2-66\,\alpha-48}{2\,(\alpha+1)^2\,(\alpha+2)^2}\end{aligned} \quad (6.18)$$

Now we obtain the third raw moment of the transmuted skew exponential distribution as

$$\begin{aligned}E(X^3) &= -\frac{(\alpha+1)\,(21\,\alpha^8+183\,\alpha^7+612\,\alpha^6+882\,\alpha^5+141\,\alpha^4-1272\,\alpha^3-1704\,\alpha^2-768\,\alpha)}{\alpha\,(4\,\alpha^7+40\,\alpha^6+168\,\alpha^5+384\,\alpha^4+516\,\alpha^3+408\,\alpha^2+176\,\alpha+32)}\\ &= -\frac{3\,(7\,\alpha^7+61\,\alpha^6+204\,\alpha^5+294\,\alpha^4+47\,\alpha^3-424\,\alpha^2-568\,\alpha-256)}{4\,(\alpha+1)^3\,(\alpha+2)^3}\end{aligned} \quad (6.19)$$

Finally the fourth raw moment has been obtained as

$$\begin{aligned}E(X^4) &= -3\,(15\,\alpha^9+178\,\alpha^8+889\,\alpha^7+2372\,\alpha^6+3401\,\alpha^5+1714\,\alpha^4-2410\,\alpha^3\\ &\quad -5120\,\alpha^2-4000\,\alpha-1280)\,(2\,(\alpha+1)^4\,(\alpha+2)^4)^{-1}\end{aligned} \quad (6.20)$$

Variance of the pdf (5.15) for $\alpha\neq 1$ is given by

$$\text{Var}(X)=-\frac{\alpha^6+8\,\alpha^5+23\,\alpha^4+18\,\alpha^3-23\,\alpha^2-52\,\alpha-32}{4\,(\alpha+1)^2\,(\alpha+2)^2} \quad (6.21)$$

6.1 Estimation

Here, we consider estimation by the method of moments. Let $x_1, \ldots, x_n$ be a random sample from the Eq. (5.15) for $\alpha \neq 1$. For the moment estimation, let $m_1 = \frac{1}{n}\sum_{j=1}^{n} x_j$ and $m_2 = \frac{1}{n}\sum_{j=1}^{n} x_j^2$. By equating the theoretical moments of Eq. (5.15) for $\alpha \neq 1$ with the sample moments, one can obtain the relation

$$0 = \left(\alpha^6 + 2\,\alpha^5 - 9\,\alpha^4 - 26\,\alpha^3 + 9\,\alpha^2 + 80\,\alpha + 64\right)\,k + 6\,\alpha^5 + 32\,\alpha^4 + 44\,\alpha^3 - 32\,\alpha^2 - 132\,\alpha - 96, \quad (6.22)$$

where $k = \frac{m_2}{m_1^2}$.
The Eq. (6.22) can be solved by the method of Newton-Raphson method.

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